EXPLICIT TENSORS IN $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ OF BORDER RANK AT LEAST $2n-1$

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Abstract. For odd n, I write down tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ of border rank $2n-1$, showing the non-triviality of the Young-flattening equations of [7]. I also study the border rank of the tensors of Alexeev et. al. [1], showing the tensors $T_{2^k} \in \mathbb{C}^k \otimes \mathbb{C}^{2^k} \otimes \mathbb{C}^{2k}$, despite having rank equal to $2^{k+1}-1$, have border rank equal to 2^k, the minimum of any concise tensor.

1. Results

Let A, B, C be $\mathbb{C}^n$. Let $T \in A \otimes B \otimes C$ be a tensor, i.e., a bilinear map $A^* \times B^* \to C$. One says T has *rank one* if there exists $a \in A$, $b \in B$, and $c \in C$ such that $T(\alpha, \beta) = \alpha(a)\beta(b)c$, i.e., $T = a \otimes b \otimes c$. In this case T can be computed by performing just one scalar multiplication. More generally the *rank* of a tensor T is the smallest r such that T may be written as a sum of r rank one bilinear maps, i.e., such that there exist $a_1, \dots, a_r \in A$, $b_1, \dots, b_r \in B$, $c_1, \dots, c_r \in C$ such that $T = \sum_{j=1}^r a_j \otimes b_j \otimes c_j$. One writes $\mathbf{R}(T) = r$. The rank of a tensor is a measure of its complexity. Rank is not semi-continuous - the limit of a sequence of tensors of rank r need not have rank at most r. To rectify this situation, define the *border rank* of $T \in A \otimes B \otimes C$ to be the smallest r such that T is a limit of a sequence (possibly constant) of tensors of rank r, and write $\underline{\mathbf{R}}(T) = r$. The border rank is often used as a substitute for the rank in measuring complexity. (If one is interested in questions such as the exponent of matrix multiplication, it does not matter whether one uses rank or border rank.) From an algebraic perspective, the set of tensors of border rank at most r is the Zariski closure of the set of tensors of rank at most r and is denoted $\hat{\sigma}_r := \hat{\sigma}_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subset A \otimes B \otimes C$. (The variety $\hat{\sigma}_r$ is familiar in algebraic geometry, the cone over the r-th secant variety of the Segre variety.) The set $\hat{\sigma}_r$ is by construction an algebraic variety, and thus membership in it can in principle be tested by testing the vanishing of polynomials. To do this one must first find the relevant polynomials. Often one can describe explicit polynomials that by construction vanish on $\hat{\sigma}_r$, but a difficulty arises in showing that the polynomials do not vanish identically. This was the case for the Young flattenings of [7].

By [7] the size $\binom{n-1}{p}r+1$ minors of

$$T_A^{\wedge p} : \Lambda^p A \otimes B^* \to \Lambda^{p+1} A \otimes C, \tag{1.1}$$

where $a_1 \wedge \cdots \wedge a_p \otimes \beta \mapsto a_1 \wedge \cdots \wedge a_p \wedge T(\beta)$ (note that $T(\beta) \in A \otimes C$)), vanish on elements of $\hat{\sigma}_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ up to $r = \lceil \frac{n^2}{n-p} \rceil$. If $n = 2p+1$, $\frac{n^2}{n-p} = 2n-2+\frac{1}{p+1}$ and if $n = 2p+2$, then $\frac{n^2}{n-p} = 2n-4+\frac{4}{p+2}$, so they potentially give equations for $\hat{\sigma}_r$ up to $r = 2n-1$ when n is odd and $r = 2n-3$ when n is even. In [7] it was shown these equations are nontrivial (i.e., do not vanish identically) up to $r = 2n - \lceil \sqrt{n} \rceil$ by showing they did not vanish on the matrix multiplication tensor $M_m \in \mathbb{C}^{m^2} \otimes \mathbb{C}^{m^2} \otimes \mathbb{C}^{m^2}$, and as a result giving the bound $\underline{\mathbf{R}}(M_m) \geq 2m^2 - m$. (Here $n = m^2$.) The main purpose of this article is to close the gap:

Theorem 1.1. *When n is odd and equal to $2p+1$, the maximal minors of (1.1) give nontrivial equations for $\hat{\sigma}_r(Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$, the tensors of border rank at most r in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$,*

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up to $r = 2n-1$. These also give equations up to $r = 2n-3$ when n is even. In fact the maximal minors do not vanish on the explicit tensors $T_n(\lambda) \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ of (2.1) when n is odd, and the same tensors padded by zeros give the bound when n is even.

These are the largest values of r we know how to test for, namely tensors that fail to satisfy the equations for injectivity of (1.1).

Corollary 1.2. *Let $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$ denote the matrix multiplication operator. Then M_n satisfies the known nontrivial equations for the variety of tensors of border rank $2n^2 - n + 1$, $\hat{\sigma}_{2n^2-n+1}(Seg(\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}))$.*

What is known regarding lower bounds is $\underline{\mathbf{R}}(M_n) \geq 2n^2 - n$ [7] and $\mathbf{R}(M_n) \geq 3n^2 - 4n^{3/2} + n$ [4]. Asymptotic upper bounds on the order of $n^{2.37}$ (see e.g., [14]) are know for both. Very few concrete upper bounds are known: $\mathbf{R}(M_3) \leq 23$ [3], $\mathbf{R}(M_{2^k}) \leq 7^k$ [12], and $\underline{\mathbf{R}}(M_3) \leq 21$ [10].

In [1] (also see [13]), setting $n = 2^k$, they give an explicit sequence of tensors $T_n \in \mathbb{C}^{k+1} \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, see (3.1), of rank $2n-1$ and explicit tensors $T'_{n+1} \in \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^n$ of rank $3(n+1) - k - 4$, see §4. Their tensors may be defined over an arbitrary field.

Proposition 1.3. *Let $n = 2^k$. The tensors $T_n \in \mathbb{C}^{k+1} \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ of (3.1) have border rank n, i.e., $\underline{\mathbf{R}}(T_n) = n < \mathbf{R}(T_n) = 2n - 1$.*

Proposition 1.4. *Let $n = 2^k$. The tensors $T'_{n+1} \in \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ of §4 satisfy $n + 1 \leq \underline{\mathbf{R}}(T'_{n+1}) \leq 2(n+1) - 2 - k < \mathbf{R}(T'_{n+1}) = 3(n+1) - 4 - k$.*

The maximal rank of a tensor in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is at most n^2, although it is not known if this actually occurs. The maximal rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is $\lceil \frac{3n}{2} \rceil$ The maximal border rank of a tensor in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is $\lceil \frac{n^3}{3n-2} \rceil$ for all $n \neq 3$ and five when $n = 3$ [11, 8]. Results of Raz [9] indicate that it may be difficult to write down explicit tensors of large rank.

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2. Proof of Theorem 1.1

Let $a_{-p}, \dots, a_p$ be a basis of A, $b_1, \dots, b_n$ a basis of B, and $c_1, \dots, c_n$ a basis of C.

Let $\lambda_{i,u}$ be numbers satisfying open conditions to be specified below. (They may be chosen to be integers.) Consider

$$
\begin{aligned}
T_{n,p}(\lambda) := & a_0 \otimes (b_1 \otimes c_1 + \cdots + b_n \otimes c_n) \\
& + a_1 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3 + \cdots b_{n-1} \otimes c_n) \\
& + a_2 \otimes (b_1 \otimes c_3 + b_2 \otimes c_4 + \cdots b_{n-2} \otimes c_n) \\
& + \vdots \\
& + a_p \otimes (b_1 \otimes c_{p+1} + \cdots b_{n-p} \otimes c_n) \\
& + a_{-1} \otimes (\lambda_{1,1} b_{p+1} \otimes c_1 + \cdots + \lambda_{1,n-p} b_n \otimes c_{n-p}) \\
& + a_{-2} \otimes (\lambda_{2,1} b_{p+2} \otimes c_1 + \cdots + \lambda_{2,n-p-1} b_n \otimes c_{n-p-1}) \\
& + \vdots \\
& + a_{-p+1} \otimes (\lambda_{p-1,1} b_3 \otimes c_1 + \cdots + \lambda_{p-1,n-2} b_n \otimes c_{n-2}) \\
& + a_{-p} \otimes (\lambda_{p,1} b_2 \otimes c_1 + b_3 \otimes c_2 + \cdots \lambda_{p,n-1} b_n \otimes c_{n-1}).
\end{aligned} \tag{2.1}
$$

When $n = 2p + 1$, write $T_n(\lambda) = T_{2p+1,p}(\lambda)$.

Write $T = \sum a_j \otimes X_j$ and use $X_0 : B^* \to C$ to identify C with B^*. Then, by [4], (1.1) is injective if and only if

$$\det([X_i, X_j]) \neq 0$$

Notice that our choice of bases and X_0 gives a grading to $\mathfrak{gl}(B)$ such that $X_j \in \mathfrak{gl}(B)_j$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \in \mathfrak{g}_{i+j}$.

Thus, omitting the zero index, $[X_i, X_j]$, is an $n \times n$ matrix that is zero if i, j are greater than zero, and otherwise zero except for the $(i+j)-th$ diagonal, all of whose entries are nonzero as long as for each fixed i, the $\lambda_{i,u}$ are distinct as u varies. Writing the $2np \times 2np$ matrix, ordered $p, p-1, \ldots, -p$, as four equal square blocks, the first block is zero, and the its determinant is thus, up to sign, the square of the determinant of the lower left block. We now consider this block.

For example, when $p = 2$ and $n = 5$ we get (blocking $2, 1, -1, -2$ for both rows and columns) the lower left four blocks are:

$$[X_2, X_{-1}] = \begin{pmatrix} 0 & \lambda_{1,2} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{1,3} - \lambda_{1,1} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,2} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{1,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[X_2, X_{-2}] = \begin{pmatrix} \lambda_{2,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2,2} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{2,3} - \lambda_{2,1} & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{2,2} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{2,3} \end{pmatrix}$$

$$[X_1, X_{-1}] = \begin{pmatrix} \lambda_{1,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{1,2} - \lambda_{1,1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{1,3} - \lambda_{1,2} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,3} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{1,4} \end{pmatrix}$$

$$[X_1, X_{-2}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_{2,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2,2} - \lambda_{2,1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{2,3} - \lambda_{2,2} & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{2,31} & 0 \end{pmatrix}$$

Consider the first column of the $pn \times pn$ matrix $([X_i, X_{-k}])$, $1 \leq i, k \leq p$. All the entries are zero except the first entry of the lowest block, i.e., the entry in the slot $((p-1)n+1, 1)$, which is $\lambda_{p,1}$.

Now consider the second column. There are two nonzero entries - the first entry of the second lowest block, which is $\lambda_{p-1,2}$, and the second entry of the last block, which is $\lambda_{p,2}$. The $(n+1)$-st column (first column of the second block) also has two nonzero entries, and they occur at the same heights, the entries are respectively $\lambda_{p-1,1}$ and $\lambda_{p,1}$. Thus these two columns contribute $\det \begin{pmatrix} \lambda_{p-1,2} & \lambda_{p,2} \\ \lambda_{p-1,1} & \lambda_{p,1} \end{pmatrix}$ to the determinant.

Consider the third column. If $p > 2$, there are three nonzero entries, the first of the third to last block, the second of the second to last block, and the third of the last block. The second column of the second block and the third column of the third block all have the same nonzero entries. The result is a contribution of

$$\det \begin{pmatrix} \lambda_{p-2,3} & \lambda_{p-2,2} & \lambda_{p-2,1} \\ \lambda_{p-1,3} & \lambda_{p-1,2} & \lambda_{p-1,1} \\ \lambda_{p,3} & \lambda_{p,2} & \lambda_{p,1} \end{pmatrix}$$

to the determinant.

If $p=2$, there are two nonzero entries, and one gets a contribution of $\det\begin{pmatrix}\lambda_{1,3}-\lambda_{1,1} & \lambda_{1,2}-\lambda_{1,1}\\ \lambda_{2,3}-\lambda_{2,1} & \lambda_{2,2}-\lambda_{2,1}\end{pmatrix}$. There are three more contributions to the determinant of the lower left large block in this case, namely

$$\det\begin{pmatrix}\lambda_{1,4}-\lambda_{1,2} & -\lambda_{2,2}\\ \lambda_{1,3}-\lambda_{1,2} & \lambda_{2,3}-\lambda_{2,2}\end{pmatrix}\det\begin{pmatrix}-\lambda_{1,4} & -\lambda_{2,3}\\ \lambda_{1,4}-\lambda_{1,3} & -\lambda_{2,1}\end{pmatrix}(-\lambda_{1,4}).$$

In this case it is clear that for a general choice of $\lambda_{i,j}$ the determinant is nonzero.

In general, considering the i-th column, for $i\leq p$, there is a contribution of the determinant of an $i\times i$ matrix whose (s,t)-th entry is $\lambda_{p-i+1+s,i+t}$, or $\lambda_{p-i+1+s,i+t}-\lambda_{p-i+1+s,i+t+1}$, or $-\lambda_{p-i+1+s,i+t+1}$. For $i\geq p$ one gets $p\times p$ matrices until they start shrinking in size.

In all cases, for any given minor of size f that appears, it will have a unique term with coefficient plus or minus one on $\Pi_{s=1}^{f}\lambda_{p-i+1-s,i+s}$, so for a generic choice of $\lambda_{u,i}$ it will not vanish. Thus for a generic choice no minors will vanish, which means that their product, the determinant, will not vanish either, proving the theorem. Note that we can choose the $\lambda_{i,u}$ to be integers.

Remark 2.1. The sum of the first $p+1$ terms in $T_n(\lambda)$ has border rank n (see the discussion below), so one might obtain a sequence of higher complexity by inserting constants in these terms as was done for the last p terms. However at the moment we have no way to measure such higher complexity.

3. The tensors T_n of [1]

I restrict to the case $n=2^k$ because the other cases are similar only padded with zeros. In [1] they define tensors $T_n\in\mathbb{C}^{k+1}\otimes\mathbb{C}^n\otimes\mathbb{C}^n=A\otimes B\otimes C$ by

$$\begin{aligned}T_n :=& a_0\otimes(b_1\otimes c_1+\cdots+b_n\otimes c_n)\\ &+a_1\otimes(b_1\otimes c_n)\\ &+a_2\otimes(b_1\otimes c_{n-1}+b_2\otimes c_n)\\ &+a_3\otimes(b_1\otimes c_{n-3}+b_2\otimes c_{n-2}+b_3\otimes c_{n-1}+b_4\otimes c_n)\\ &+a_4\otimes(b_1\otimes c_{n-2^3+1}+\cdots+b_{2^3}\otimes c_n)\\ &+\vdots\\ &+a_k\otimes(b_1\otimes c_{n-2^{k-1}+1}+\cdots+b_{2^{k-1}}\otimes c_n).\end{aligned}\tag{3.1}$$

Here I have changed the indices slightly from [1].

For example, when $k=3$, in matrices, this looks like:

$$T_8(A^*)=\begin{pmatrix}a_0 &&&&&&&\\ & a_0 &&&&&&\\ && a_0 &&&&&\\ &&& a_0 &&&&\\ a_3 &&&& a_0 &&&\\ & a_3 &&&& a_0 &&\\ a_2 && a_3 &&&& a_0 &\\ a_1 & a_2 && a_3 &&&& a_0\end{pmatrix}.$$

Note that if we change bases and write the tensor as

$$
\begin{aligned}
T_n :=& a_0\otimes(\tilde b_n\otimes c_1+\cdots+\tilde b_1\otimes c_n)\\
&+a_1\otimes(\tilde b_n\otimes c_n)\\
&+a_2\otimes(\tilde b_n\otimes c_{n-1}+\tilde b_{n-1}\otimes c_n)\\
&+a_3\otimes(\tilde b_n\otimes c_{n-3}+\tilde b_{n-1}\otimes c_{n-2}+\tilde b_{n-2}\otimes c_{n-1}+\tilde b_{n-3}\otimes c_n)\\
&+a_4\otimes(\tilde b_n\otimes c_{n-2^3+1}+\cdots+\tilde b_{n-2^3+1}\otimes c_n)\\
&+\vdots\\
&+a_k\otimes(\tilde b_n\otimes c_{n-2^{k-1}+1}+\cdots+\tilde b_{n-2^{k-1}+1}\otimes c_n).
\end{aligned}
\tag{3.2}
$$

With the identification $\tilde b_j=c_j$, we obtain $T_n\in A\otimes S^2B$.

Proof of Proposition 1.3. Here is an explicit curve $T_n(t)$ such that $\lim_{t\to0}T_n(t)=T_n$. I explain below how it was found. I use (3.2) with $\tilde b_j=c_j$ to obtain a partially symmetric tensor.

$$
\begin{aligned}
&T_n(t)=\\
&\frac{1}{t^{n-1}}\{(a_0+t^{n-2^{k-1}}a_k+t^{n-2^{k-2}}a_{k-1}+\cdots+t^{n-1}a_1)\otimes(c_n+tc_{n-1}+t^2c_{n-2}+\cdots+t^{n-1}c_1)^{\otimes2}\\
&+(-a_0+t^{n-2^{k-1}}a_k+t^{n-2^{k-2}}a_{k-1}+\cdots-t^{n-2}a_2)\otimes(-c_n+tc_{n-1}-t^2c_{n-2}+\cdots+t^{n-3}c_3-t^{n-2}c_2)^{\otimes2}\\
&+(a_0-t^{n-2^{k-1}}a_k-t^{n-2^{k-2}}a_{k-1}+\cdots+t^{n-4}a_3)\otimes(c_n-tc_{n-1}+t^2c_{n-2}+\cdots+t^{n-4}c_4-t^{n-3}c_3)^{\otimes2}\\
&+\vdots\\
&-a_0\otimes c_n\otimes c_n\}
\end{aligned}
\tag{3.3}
$$

where each term has the highest power of t occurring to one power less than the previous. Except for the first term, the signs of the coefficients are determined by parity - all even exponents have the same sign in each expression, as do all odd. $\square$

To see how to deduce the result without an explicit curve, write $T_n=\sum a_j\otimes X_j$. Use X_0 to identify $C\simeq B^*$, then $[X_i,X_j]=0$ for all $0\le i,j\le k$. In other words, using a_0 to identify $C\simeq B^*$, $T(A^*)\subset\mathfrak{gl}(B)$ is an abelian sub-algebra. If this abelian sub-algebra is also diagonalizable, or the limit of a sequence of diagonalizable subalgebras, then $\underline{\mathbf{R}}(T_n)=n$, see [5, §7.6.5]. Let $Red(n)\subset G(n,\mathfrak{gl}(n))$ denote the Zariski closure of the set of diagonalizable subspaces, where $G(n,\mathfrak{gl}(n))$ denotes the Grassmannian of n-planes in $\mathfrak{gl}(n)$, the space of endomorphisms of $\mathbb{C}^n$. As observed in [6], Strassen's equations detect abelian subspaces of $G(n,\mathfrak{gl}(n))$ and a tensor will have border rank n if it is a limit of diagonalizable subspaces, i.e., if it is an element of $Red(n)$. Similarly, a concise tensor $T\in\mathbb{C}^k\otimes\mathbb{C}^n\otimes\mathbb{C}^n$ satisfying Strassen's equations will have border rank n if $T(\mathbb{C}^{k*})\subset\mathfrak{gl}(n)$ is contained in a subspace of an element of $Red(n)$. One can deduce Proposition 1.3 immediately at this point because, as L. Manivel (personal communication) observes, $Red(n)$ always contains the centralizers of regular elements, in particular regular nilpotent elements. The images $T_n(\mathbb{C}^{k^*})$ always are contained in the centralizer of the regular nilpotent element with 1's below the diagonal and zero's elsewhere. (This centralizer consists of lower triangular matrices whose entries are constant on the band diagonals.)

The explicit curve was found as follows: In general, for a projective variety $X\subset\mathbb{P}V$, let $x\in X$ be a point such that the only nonzero projective differential invariants are the projective fundamental forms (see [2, §2] for the definition of projective differential invariants and more

detail regarding this construction). Then by taking $r-1$ curves limiting to X, one obtains a point of $\sigma_r(X)$ of the following form: choose a splitting of the osculating sequence $V = \hat{x}\oplus T\oplus N_2\oplus\cdots\oplus N_f$, so the k-th fundamental form may be identified with a map $\mathbb{F}_k : S^kT \to N_k$. Then the points are obtained by choosing $r-1$ vectors $v_1,\dots,v_{r-1}\in T$ and the limiting point is

$$\sum_{k=1}^{r-1}\sum_{i_1+\cdots i_k=r-1}\mathbb{F}_k(v_{i_1}\cdots v_{i_k}) = v_{r-1} + \sum_{j=1}^{\lfloor\frac{r-1}{2}\rfloor}\mathbb{F}_2(v_{r-j}v_j)+\cdots$$

When X is the triple Segre product these tensors look like:

$$\begin{array}{cr}
r & T\\
1 & a_1\otimes b_1\otimes c_1\\
2 & a_1\otimes b_1\otimes c_2 + a_1\otimes b_2\otimes c_1 + a_2\otimes b_1\otimes c_1\\
3 & a_1\otimes b_2\otimes c_2 + a_2\otimes b_2\otimes c_1 + a_2\otimes b_1\otimes c_2 + a_1\otimes b_1\otimes c_3 + a_1\otimes b_3\otimes c_1 + a_3\otimes b_1\otimes c_1\\
4 & a_2\otimes b_2\otimes c_2 + \displaystyle\sum_{\sigma\in\mathfrak{S}_3} a_{\sigma(1)}\otimes b_{\sigma(2)}\otimes c_{\sigma(3)} + a_4\otimes b_1\otimes c_1 + a_1\otimes b_4\otimes c_1 + a_1\otimes b_1\otimes c_4\\
\vdots &
\end{array}$$

In terms of $T(A^*)$, one gets

$$\begin{pmatrix}
a^n & a^{n-1} & a^{n-2} & \cdots & a^2 & a^1\\
a^{n-1} & a^{n-2} & a^{n-3} & \cdots & a^1 & 0\\
a^{n-2} & a^{n-3} & \cdots & a^1 & 0 & 0\\
 & & \vdots & & & \\
a^2 & a^1 & 0 & & & \\
a^1 & 0 & & \cdots & & 0
\end{pmatrix}$$

If in the derivatives, we set the terms corresponding to a_j all equal to zero except for $j = n, n-1, n-3, n-7 = n-2^3+1, \dots, n-2^s+1, .., 1 = n-2^k+1$, and then us the symmetric form for T_n, we obtain T_n.

4. The tensors T'_{n+1} of [1]

In [1], they also define tensors in $\mathbb{C}^{n+1}\otimes\mathbb{C}^n\otimes\mathbb{C}^{n+1}$ by enlarging the matrices to have size $n+1$ and adding vectors with a single nonzero entry in the last column. For example, when $k=3$ (so $n=8$), one gets

$$T'_9(A^*) := \begin{pmatrix}
a_0 & & & & & & & & a_4\\
 & a_0 & & & & & & & a_5\\
 & & a_0 & & & & & & a_6\\
 & & & a_0 & & & & & a_7\\
a_3 & & & & a_0 & & & & a_8\\
 & a_3 & & & & a_0 & & & \\
a_2 & & a_3 & & & & a_0 & & \\
a_1 & a_2 & & a_3 & & & & a_0 &
\end{pmatrix}$$

which they express as a tensor in $\mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}$ by adding zeros. These tensors have rank close to $3n$, to be precise $\mathbf{R}(T'_n) = 3n - 2H(n-1) - \lfloor\log_2(n-1)\rfloor - 2$, where $H(m)$ is the number of 1's in the binary expansion of m, so the rank is best if $n-1 = 2^k$, in which case $\mathbf{R}(T'_{2^k+1}) = 3(2^k+1) - 4 - k$. The border rank is smaller. Write $T'_{n+1} = T_n + T''_n$ where

$T''_n = (a_{k+1}\otimes b_1 + a_{k+2}\otimes b_2 + \cdots + a_n\otimes b_{n-k})\otimes c_{n+1}$. Thus $\underline{\mathbf{R}}(T'_{n+1}) \leq \underline{\mathbf{R}}(T_n)+\underline{\mathbf{R}}(T''_n) = n+n-k$. I expect the actual border rank to be lower.

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